Lecture 08: Shamir Secret Sharing (Introduction)

Objective

- The objective of this new cryptographic primitive is to share a secret s among n people such that the following holds. For a fixed number $t < n$, the following conditions are satisfied.
  - If $< t$ parties get together, then they get no additional information about the secret.
  - If $\geqslant t$ parties get together, then they can correctly reconstruct the secret.
- In this lecture, we study an introductory version of this cryptographic primitive

Learning Arithmetic Over $(\mathbb{Z}_p, +, \times)$ I

- We have seen that $(\mathbb{Z}_p, +, \times)$ is a field, when p is a prime
  - Recall that $+$ is integer additional modulo the prime p
  - Recall that $\cdot$ is integer multiplication modulo the prime p
  - For example, the additive inverse of x is $(p - x)$, for $x \in \mathbb{Z}_p$ (because $x + (p - x) = 0 \mod p$)
  - In the homework you have shown that the multiplicative inverse of x is x^{p-2}, for $x \in \mathbb{Z}_p^*$ (i.e., $x \times (x^{p-2}) = 1 \mod p$)

For a working example suppose $p = 5$. Therefore, $x^{p-2} = x^3$ is the multiplicative inverse of x in $(\mathbb{Z}_5, +, \times)$

- The multiplicative inverse of 1 is $1^{p-2} = 1$, i.e. $(1/1) = 1$
- The multiplicative inverse of 2 is $2^{p-2} = 2 \times 2 \times 2 = 4 \times 2 = 3$, i.e. $(1/2) = 3$
- The multiplicative inverse of 3 is $3^{p-2} = 3 \times 3 \times 3 = 4 \times 3 = 2$, i.e. $(1/3) = 2$
- The multiplicative inverse of 4 is $4^{p-2} = 4 \times 4 \times 4 = 1 \times 4 = 4$, i.e. $(1/4) = 4$

Interpreting "fractions" over the field $(\mathbb{Z}_p, +, \times)$

- When we write $4/3$
- We mean $4 \cdot (1/3)$,
- That is 4 multiplied by the "multiplicative inverse of 3"
- That is 4 multiplied by 2 (because in the previous slide we saw that the multiplicative inverse of 3 in $(\mathbb{Z}_5, +, \times)$ is 2)
- The answer, therefore, is 3 (because $4 \times 2 = 3 \mod 5$)

Note

While working over real numbers we associate "4/3" to the fraction "$1.333\cdots$," i.e. a fractional number. But when working over the field $(\mathbb{Z}_p, +, \times)$ we will interpret the expression "4/3" as the number "$4 \times$ mult-inv(3)"

Experiments

Coding Exercise

Students are highly encouraged to go to cocalc.com and explore field arithmetic using sage

Secret Sharing: Goal (Introduction)

- Suppose a central authority P has a secret s (some natural number)
- The central authority wants to share the secret among n parties $P_1, P_2, \ldots, P_n$ such that
  - **Privacy.** No party can reconstruct the secret s.
  - **Reconstruction.** Any two parties can reconstruct the entire secret s

Secret Sharing: Algorithms (Introduction)

Sharing Algorithm: SecretShare (s, n).

- Takes as input a secret s
- Takes as input n, the number of shares it needs to create
- Outputs a vector $(s_1, s_2, \ldots, s_n)$ the *secret shares* for each party

Reconstruction Algorithm: SecretReconstruct $(i_1, s^{(1)}, i_2, s^{(2)})$.

- Takes as input the identity i of the first party and identity j of the second party
- Takes as input their respective secrets $s^{(1)}$ and $s^{(2)}$
- Outputs the reconstructed secret $\widetilde{s}$
- The probability that the reconstructed secret $\widetilde{s}$ is identical to the original secret s is close to 1

Example: Shamir's Secret Sharing Scheme (Introduction) I

Intuition underlying the construction:

- Given one point in a plane, there are a lot of straight lines passing through it (In fact, we need the fact that *every* length of the intercept on the Y-axis is equally likely)
- But, given two points in a plane, there is a *unique* line passing through it, thus the length of the intercept on the Y-axis is unique

Example: Shamir's Secret Sharing Scheme (Introduction) II

Let $(\mathbb{F}, +, \times)$ be a field such that $\{0, 1, \ldots, n\} \subseteq \mathbb{F}$ and the secret $s \in \mathbb{F}$. The secret sharing algorithm is provided below.
SecretShare (s, n).

- Choose a random line $\ell(X)$ passing through the point $(0, s)$. Note that the equation of the line is $a \cdot X + s$, where a is randomly chosen from $\mathbb{F}$
- Evaluate the line $\ell(X)$ at $X = 1$, $X = 2$, ..., $X = n$ to generate the secret shares s_1, s_2, ..., s_n. That is, $s_1 = \ell(X = 1)$, $s_2 = \ell(X = 2)$, ..., $s_n = \ell(X = n)$

Example: Shamir's Secret Sharing Scheme (Introduction) III

The reconstruction algorithm is provided below.
SecretReconstruct $(i_1, s^{(1)}, i_2, s^{(2)})$.

- Compute the equation of the line

$$\ell'(X) := \frac{s^{(2)} - s^{(1)}}{i_2 - i_1} \cdot X + \left(\frac{i_2 s^{(1)} - i_1 s^{(2)}}{i_2 - i_1} \right)$$

- Let $\widetilde{s}$ be the evaluation of the line $\ell'(X)$ at $X = 0$. That is, return $\widetilde{s} = \ell'(0) = \left(\frac{i_2 s^{(1)} - i_1 s^{(2)}}{i_2 - i_1} \right)$.

Example: Shamir's Secret Sharing Scheme (Introduction) IV

Privacy Argument

- Given the share of only one party $(i_1, s^{(1)})$, there is a unique line passing through the points $(i_1, s^{(1)})$ and $(0, \alpha)$, for every $\alpha \in \mathbb{F}$.
- So, *all secrets are equally likely from this party's perspective*

In the future, we will mathematically formalize and prove the *italicized* statement above

An Illustrative Example I

- Suppose yesterday morning the central authority P gets the secret $s = 3$
- And the central authority wants to share the secret among $n = 4$ parties

- Note that we can work over $(\mathbb{Z}_p, +, \times)$, where $p = 5$
  - Because $\{1, \ldots, 4\} \subseteq \mathbb{Z}_p^*$

An Illustrative Example II

Execution of the Secret-sharing Algorithm

- The central authority picks a random line that passes through $(0, s) = (0, 3)$
- The equation of such a line looks like

$$\ell(X) = k \cdot X + 3,$$

  where k is an element in $\mathbb{Z}_p$ chosen uniformly at random
- Suppose it turns out that $k = 2$
- Now, the share of the four parties are evaluation of the line $\ell(X)$ at $X = 1$, $X = 2$, $X = 3$, and $X = 4$.
- So, the secret shares of parties 1, 2, 3, and 4 are respectively

$$\begin{aligned} s_1 &= \ell(X = 1) = 2 \times 1 + 3 = 0 \\ s_2 &= \ell(X = 2) = 2 \times 2 + 3 = 2 \\ s_3 &= \ell(X = 3) = 2 \times 3 + 3 = 4 \\ s_4 &= \ell(X = 4) = 2 \times 4 + 3 = 1 \end{aligned}$$

An Illustrative Example III

- Yesterday, at the end of the day, the central authority provides each party their respective secret share (that is, the central authority provides 0 to party 1, 2 to party 2, 4 to party 3, and 1 to party 4)
  - Note that the equation of the line $\ell(X)$ is hidden from the parties
  - All that the party i knows is that the line $\ell(X)$ passes through the point (i, s_i)
- After that, the parties 1, 2, 3, and 4 part ways and go their own homes

An Illustrative Example IV

Today, let us zoom into party 3's home

- Party 3 has secret share 4
- To find the secret s, party 3 enumerates all lines passing through the point $(3, 4)$

$$\begin{aligned}
\ell_0(X) &= 0 \cdot X + 4 \\
\ell_1(X) &= 1 \cdot X + 1 \\
\ell_2(X) &= 2 \cdot X + 3 \\
\ell_3(X) &= 3 \cdot X + 0 \\
\ell_4(X) &= 4 \cdot X + 2
\end{aligned}$$

An Illustrative Example V

- Note that the central authority could have picked up *any* of these lines yesterday
- Note that
  - The line ℓ_0 has intercept 4 on the Y-axis (i.e., the evaluation of the line at $X = 0$),
  - The line ℓ_1 has intercept 1 on the Y-axis,
  - The line ℓ_2 has intercept 3 on the Y-axis,
  - The line ℓ_3 has intercept 0 on the Y axis, and
  - The line ℓ_4 has intercept 2 on the Y-axis
- So, it is equally likely that the central authority shared the secret 0, 1, 2, 3, or 4 yesterday

An Illustrative Example VI

Tomorrow, party 3 decides to meet party 1 and they will together work on reconstructing the secret. Their reconstruction steps are provided below.

- Party 1's secret share is 0, and party 3's secret share is 4
- So, the line has to pass through the points $(1, 0)$ and $(3, 4)$
- The slope of the line is

$$\begin{aligned} \frac{4-0}{3-1} &= 4 \times (1/2) \\ &= 4 \times 3, \qquad \text{because the multiplicative inverse of 2 is 3} \\ &= 2 \end{aligned}$$

- So, the equation of the line is of the form

$$\ell'(X) = 2 \cdot X + c$$

- And, at $X = 1$ the line evaluates to 0. So, the line is $\ell'(X) = 2 \cdot X + 3$

An Illustrative Example VII

- Note that the reconstructed line is identical to the line used by the central authority!
- The intercept of the line $\ell'(X)$ on the Y-axis is $\widetilde{s} = \ell'(X = 0) = 3$, which is identical to the secret shared by the central authority!

Generalization

In the next lecture, we will see how to generalize this construction so that we can ensure that any t parties can recover the secret, and no $(t-1)$ parties can recover the secret, where $t \in \{2, \ldots, p-1\}$